DIGITAL SPECTRAL ANALYSIS
WITH APPLICATIONS
S.LAWRENCE MARPLE, JR.

SUMMARY

This new book provides a broad perspective of spectral estimation techniques and their implementation. It concerned with spectral estimation of discrete-space sequences derived by sampling continuous-space signals. Among its key features, the book:

· Emphasizes the behavior of each spectral estimator for short data records.
· Provides 35 computer programs, including fast algorithms.
· Provides the theoretical background and review material in linear systems, Fourier transforms matrix algebra, random processes, and statics.
· Summarizes classical spectral estimation as it is practiced today.
· Covers Prony’s method, parametric methods, the minimum variance method, eigenanalysis-based estimators, multichannel methods, and two-dimensional methods.
· Includes problems.
· Contains appendices that cover Sunspot Numbers, Complex Test Data, Temperature Data, and Program Conversion for Complex-to-Real Case.

Of Special Interest
A disk is included that has a double-sides 360kB format readable by any personal computer with an MS-DOS 2 or 3 operating system, such as the IBM XT or AT.

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